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the Pn and Sn Methods are Equivalent

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Summary

In September 2001 I published a report [1] demonstrating that the P_n and S_n, or rather S_{n+1}, methods are equivalent, in the sense that the basic assumption of both methods defines an approximate solution for moments of the particle flux as a Gaussian quadrature over the cosine range (-1,+1), using a Legendre series expansion. It is important to understand that these methods do not attempt to directly define the **angular flux**, which is a continuous function of angle; they both only define an approximation to the **Legendre moments of the flux**, as a discrete quadrature; they define moments such as scalar flux (to define reactions, etc. within an object) and scalar current (to define leakage from the object). In the lion's share of our applications these moments are sufficient to define the solutions we require, which is why these methods are so widely used. Today's report is merely a summary of my 2001 report, but I hope it will inspire readers to also read the 2001 report [1]. For simplicity in today's report the notation and equation I use refer only to the simplest cases, planar geometry, but the conclusions are actually very general.

Unfortunately, over twenty (20) year later I still find reports and textbooks [2] that fail to recognize this equivalence. Today far too many sources still claim that the P_n method represents the angular flux as an n-th order Legendre expansion. **THIS IS INCORRECT; this method does not approximate the angular flux** What this method actually approximates are the **Legendre moments of the flux**. The only assumption is that the (n+1)-th Legendre moments is zero. It makes no assumptions about the Legendre moments (n+2), (n+3),…to (infinity). **In other words, the approximation is an infinite series in which only the (n+1)-th Legendre moment is assumed to be zero.**

This seems like an innocent enough assumption. But think about it: physically how can the (n+1)-th Legendre coefficient of the angular flux be zero at all spatial points, energies, and times? The only way that this can occur is if the "particles" are constrained to travel only in discrete directions corresponding to the zeros of $P_{n+1}(\mu)$. Since the Legendre polynomials are a complete, orthogonal set, physically there is no other way that this can occur. Once you accept this fact you can start to

understand why the P_n and S_n, or rather S_{n+1}, methods are equivalent. It is because both constrain the particles to move in the discrete angular directions corresponding to the zeros of the Legendre polynomial $P_{n+1}(\mu)$. Hopefully, this can help you understand why two such seemingly different approximations lead to the same conclusion: How can the P_n Legendre expansion lead to the S_{n+1} transport in discrete directions?

Although these methods do not explicitly need or use the angular distribution we can symbolically define the angular distribution as,

$$\begin{array}{cccc} & P_n & & S_{n+1} \\ F(Z,\mu) = & \sum_{j=0}^{\infty} \frac{2j+1}{2} P_j(\mu) F_j(Z) & = & \sum_{l=1}^{n+1} G_l(Z)\,\delta[\mu - \mu_l] \end{array}$$

$$P_{n+1}(\mu_l) = 0,\ l=1 \text{ to } n+1$$

Which would appear to be nonsense, since we know the angular distribution is a continuous function of direction, not as these equations would imply as only zero along (n+1) discrete directions (S_{n+1}) or as a wildly oscillating Legendre series (P_n). Fortunately, $F(Z, \mu)$ is not used by P_n and S_{n+1}methods.

Multiplying by $P_k(\mu)$ and integrating over all directions we obtain a definition of an infinite number of Legendre coefficients, $F_k(Z)$, k = 0 to infinity, in terms of the n+1 $G_l(Z)$, l = 1 to (n+1), terms of the S_n solution,

$$\begin{array}{ccc} P_n & & S_{n+1} \\ F_k(Z) & = & \sum_{l=1}^{n+1} G_l(Z) \int_{-1}^{+1} P_k(\mu)\,\delta[\mu - \mu_l]\,d\mu \end{array}$$

$$F_k(Z) = \sum_{l=1}^{n+1} G_l(Z)\,P_k(\mu_l)$$

Rather than the **angular flux**, $F(Z, \mu)$, it is these **Legendre moments,** $F_k(Z)$ that both P_n and S_{n+1} methods approximate; both use a Gaussian Legendre quadrature expansion over the cosine range (-1, +1). The important point to note is that for k = (n+1) the discrete directions, $\delta[\mu - \mu_l]$, correspond to the zeros of P_{n+1}: k = (n+1), $P_k(\mu_l) = 0$, so that $F_k(Z) = 0$; the (n+1)-th Legendre moments of the flux is zero, for both P_n and S_{n+1} = the basic assumption of both methods.

This concludes my summary, because at this point the two methods diverge. The Pn method has a unique analytical solution, as the sum of exponential variation in space. The Sn method introduces an additional assumption as to spatial variation, usually linear. I encourage readers to next read my 2001 report [1], which discusses this topic in more detail. For example, now that we know that the solutions are the same, and we know that the unique analytical solution of the Pn method is a sum of exponentials, so is the solution of the Sn method. This suggests that rather than assuming linear variation in space, as Sn does, there may be an advantage to assuming exponential variation, as the NIOBE method does [3]. For more on this topic please see my 2001 report [1]. Also, there is the question of convergence as we increase n. Not covered so far is what does it mean for the many Sn reports/papers that do not use Gaussian Legendre discreate directions.

References

[1] “Why are the Pn and Sn Methods Equivalent/”, by D.E.Cullen, UCRL-ID-145518, Sept. 2001, Lawrence Livermore National Laboratory.

[2] There are so many textbooks that make the wrong assumption about the spherical harmonics, P_n, method, that it does not seem fair for me to reference any one textbook to illustrate this error.

[3] “A program for the Numerical Integration of the Boltzmann Transport Equation – Niobe,” by S. Preiser, ARL 60-314 (Dec. 1960).